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Matrix equation and its application to classification of finite-dimensional estimation algebras*

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Abstract Under a very natural condition, a matrix equation is proved to have only trivial solution. This result is then applied to the classification problem of finite-dimensional estimation algebras, which gives a simpler proof of Tang's recent result on the constant structure of the Ω-matrix.

Keywords: **finite-dimensional, estimation algebras, cyclic condition, constant structure of the Ω-matrix.**

1 Main Theorem

The concept of estimation algebras has been proved to be an invaluable tool in the study of nonlinear filtering problems. Wong[1] introduced the concept of Ω-matrix, whose (i, j) entry is $\omega_{ij} = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$, where f is the drift term of the state evolution equation. Yau[2] gave a complete classification of the finite-dimensional estimation algebras of maximal rank under the condition that all the entries in Ω are constants. So the difficulty of the classification problem is turned to proving the constant structure of the Ω-matrix. Chen *et al*.[3] proved that if the estimation algebra is finite-dimensional and is of maximal rank, then all the entries in Ω are degree one polynomials. Chen et al.[4] introduced a new matrix equation and showed that it has only trivial solution when the dimension of the state space is less than or equal to 4. This matrix equation plays an important role in the classification problem, but it is still not sure whether the matrix equation has only trivial solution for arbitrary state space dimension.

In this paper, under a naturally satisfied condition in the classification problem, we prove that the assertion of only trivial solution is true for arbitrary state space dimension. Our main theorem is stated as follows.

Main Theorem. *Suppose that η_4 is a homogeneous polynomial of degree four in n variables $x_1, \cdots, x_n$, and Δ is an antisymmetric matrix whose each entry is a homogeneous polynomial of degree one also in these n variables $x_1, \cdots, x_n$ such that*

$$\Delta\Delta^T = \frac{1}{2} H(\eta_4), \tag{1}$$

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where $H(\eta_4) = \left(\frac{\partial^2 \eta_4}{\partial x_i \partial x_j}\right)$ *is the Hessian matrix of* η_4.

Write

$$\Delta = \sum_{i=1}^{n} A_i x_i ,$$

where the A_i *are real antisymmetric matrices. Suppose the following cyclic condition:*

$$A_i(r,l) + A_l(i,r) + A_r(l,i) = 0, \qquad \forall\, i, r, l = 1, \cdots, n \tag{2}$$

is valid, then $\Delta = 0$.

Using the same method we can also get a corollary in the situation where all the entries in the matrix $\Delta\Delta^{\mathrm{T}} - \frac{1}{2}H(\eta_4)$ are homogeneous polynomials of degree two in $k\,(k < n)$ variables $x_1, \cdots, x_k$. Using this corollary we obtain a simpler proof of Tang's recent result[5] that ω_{ij} are constants for $k + 1 \leqslant i, j \leqslant n$, where k is the quadratic rank of the estimation algebra.

Tang was the first to prove the above result about the constant structure of Ω by introducing a series of new computations about the estimation algebras. Our present work is inspired by him, but the method of our proof is more essential.

For the purpose of convenience and brevity, we omit the preliminary section and readers can refer to ref. [5] for the background, notations and existing results of the classification problem of finite-dimensional estimation algebras for nonlinear filtering system.

2 Proof of Main Theorem and its application

Firstly we prove a lemma about the indices-permuting principle of the Hessian matrix $\frac{1}{2}H(\eta_4)$.

Lemma 1. *Suppose that* η_4 *is a homogeneous polynomial of degree* 4 *in n variables* $x_1, \cdots, x_n$, *and* $H(\eta_4) = \left(\frac{\partial^2 \eta_4}{\partial x_i \partial x_j}\right)$ *is the Hessian matrix of* η_4. *Write*

$$\frac{1}{2}H(\eta_4) = \sum_{i,j=1}^{n} H_{ij} x_i x_j ,$$

where the H_{ij} *are real symmetric matrices and* $H_{ij} = H_{ji}$. *Then the four indices of* $H_{ij}(r,l)$ *are permutable without changing the value,* $\forall\, 1 \leqslant i, j, r, l \leqslant n$.

Proof.

$$\frac{1}{2}H(\eta_4) = \sum_{i,j=1}^{n} H_{ij} x_i x_j = \sum_{i=1}^{n} H_{ii} x_i^2 + 2\sum_{i<j} H_{ij} x_i x_j .$$

Then

$$\frac{1}{2}\frac{\partial^2 \eta_4}{\partial x_r \partial x_l} = \frac{1}{2}H(\eta_4)(r,l) = \sum_{i=1}^{n} H_{ii}(r,l)\, x_i^2 + 2\sum_{i<j} H_{ij}(r,l)\, x_i x_j, \ \forall\, 1 \leqslant r, l \leqslant n.$$

So by differentiation, we have

$$H_{ii}(r,l) = \frac{1}{4}\frac{\partial^4 \eta_4}{\partial^2 x_i \partial x_r \partial x_l}, \qquad \text{for } 1 \leqslant i \leqslant n,$$

$$H_{ij}(r,l) = \frac{1}{4}\frac{\partial^4 \eta_4}{\partial x_i \partial x_j \partial x_r \partial x_l}, \qquad \text{for } i \neq j.$$

Combining these, we have

$$H_{ij}(r,l) = \frac{1}{4}\frac{\partial^4 \eta_4}{\partial x_i \partial x_j \partial x_r \partial x_l}, \qquad \forall\, i,j,r,l.$$

Noting that the order of the mixed partial differentiation can be changed, we get the indices-permuting principle, namely, the four indices of $H_{ij}(r,l)$ are permutable without changing the value, $\forall\, 1 \leqslant i, j, r, l \leqslant n$.

Proof of Main Theorem. Write

$$\frac{1}{2}H(\eta_4) = \sum_{i,j=1}^{n} H_{ij} x_i x_j,$$

where the H_{ij} are real symmetric matrices and $H_{ij} = H_{ji}$.

Then (1) implies

$$A_i A_j + A_j A_i = -2H_{ij}, \qquad A_i^2 = -H_{ii}.$$

We have

$$A_j^2(j,l) = \sum_{i=1}^{n} A_j(j,i)A_j(i,l) = -H_{jj}(j,l) = -H_{jl}(j,j) = \sum_{i=1}^{n} A_j(j,i)A_l(i,j). \quad (3)$$

Meanwhile because of the cyclic condition (2), we have

$$A_j(i,l) = A_l(i,j) + A_i(j,l).$$

So

$$A_j^2(j,l) = \sum_{i=1}^{n} A_j(j,i)A_j(i,l) = \sum_{i=1}^{n} A_j(j,i)A_l(i,j) + \sum_{i=1}^{n} A_j(j,i)A_i(j,l). \quad (4)$$

Comparing (3) with (4), we have

$$\sum_{i=1}^{n} A_j(j,i)A_i(j,l) = 0. \quad (5)$$

Now we get

$$\sum_{i,r=1}^{n} A_j(j,i)A_i(j,r)A_j(r,l) = \sum_{r=1}^{n}\Big(\sum_{i=1}^{n} A_j(j,i)A_i(j,r)\Big)A_j(r,l) = 0. \tag{6}$$

On the other hand,

$$\sum_{r=1}^{n} A_i(j,r)A_j(r,l) = -2H_{ij}(j,l) - \sum_{r=1}^{n} A_j(j,r)A_i(r,l).$$

Using the indices-permuting principle of $\frac{1}{2}H(\eta_4)$, we have the following two equations:

$$\begin{aligned}\sum_{r=1}^{n} A_i(j,r)A_j(r,l) &= -2H_{lj}(i,j) - \sum_{r=1}^{n} A_j(j,r)A_i(r,l)\\ &= \sum_{r=1}^{n} A_l(i,r)A_j(r,j) + \sum_{r=1}^{n} A_j(i,r)A_l(r,j) - \sum_{r=1}^{n} A_j(j,r)A_i(r,l),\end{aligned} \tag{7}$$

$$\begin{aligned}\sum_{r=1}^{n} A_i(j,r)A_j(r,l) &= -2H_{jj}(i,l) - \sum_{r=1}^{n} A_j(j,r)A_i(r,l)\\ &= 2\sum_{r=1}^{n} A_j(i,r)A_j(r,l) - \sum_{r=1}^{n} A_j(j,r)A_i(r,l).\end{aligned} \tag{8}$$

Formula (7) implies

$$\begin{aligned}&\sum_{i,r=1}^{n} A_j(j,i)A_i(j,r)A_j(r,l) = \sum_{i=1}^{n} A_j(j,i)\Big(\sum_{r=1}^{n} A_i(j,r)A_j(r,l)\Big)\\ &= \sum_{i,r=1}^{n}[A_j(j,i)A_l(i,r)A_j(r,j) + A_j(j,i)A_j(i,r)A_l(r,j) - A_j(j,i)A_j(j,r)A_i(r,l)]\\ &= A_jA_lA_j(j,j) + A_j^2A_l(j,j) - \sum_{i,r=1}^{n} A_j(j,i)A_i(l,r)A_j(r,j).\end{aligned}$$

Because $A_jA_lA_j$ is also antisymmetric, from (6) we get

$$\sum_{i,r=1}^{n} A_j(j,i)A_i(l,r)A_j(r,j) = A_j^2A_l(j,j). \tag{9}$$

We also have

$$\begin{aligned}&\sum_{i,r,l=1}^{n} A_j(j,i)A_i(j,r)A_j(r,l)A_j(l,j)\\ &= \sum_{r,l=1}^{n}\Big(\sum_{i=1}^{n} A_j(j,i)A_i(j,r)\Big)A_j(r,l)A_j(l,j) = 0.\end{aligned} \tag{10}$$

From (8), we can get

$$\sum_{i,r,l=1}^{n} A_j(j,i)A_i(j,r)A_j(r,l)A_j(l,j) = \sum_{i,l=1}^{n} A_j(j,i)\Big(\sum_{r=1}^{n} A_i(j,r)A_j(r,l)\Big)A_j(l,j)$$

$$= 2\sum_{i,r,l=1}^{n} A_j(j,i)A_j(i,r)A_j(r,l)A_j(l,j) - \sum_{i,r,l=1}^{n} A_j(j,i)A_j(j,r)A_i(r,l)A_j(l,j)$$

$$= 2A_j^4(j,j) - \sum_{r=1}^{n}\Big(\sum_{i,l=1}^{n} A_j(j,i)A_i(r,l)A_j(l,j)\Big)A_j(j,r).$$

From (10), (9) and (5), we have

$$2A_j^4(j,j) = \sum_{r=1}^{n} A_j^2 A_r(j,j)A_j(j,r) = \sum_{i,r=1}^{n} A_j^2(j,i)A_r(i,j)A_j(j,r)$$

$$= -\sum_{i=1}^{n} A_j^2(j,i)\Big(\sum_{r=1}^{n} A_j(j,r)A_r(j,i)\Big) = 0,$$

which is

$$A_j^4(j,j) = 0. \tag{11}$$

From the symmetry of A_j^2, we get

$$0 = A_j^4(j,j) = \sum_{i=1}^{n} A_j^2(j,i)\ A_j^2(i,j) = \sum_{i=1}^{n}[A_j^2(j,i)]^2,$$

which implies $A_j^2(j,i) = 0$. In particular, $A_j^2(j,j) = 0$.

In view of the antisymmetry of A_j, we have

$$0 = A_j^2(j,j) = \sum_{i=1}^{n} A_j(j,i)\ A_j(i,j) = -\sum_{i=1}^{n}[A_j(j,i)]^2,$$

which immediately implies

$$A_j(j,i) = 0. \tag{12}$$

As a special case of the indices-permuting principle, we have

$$H_{pp}(j,j) = H_{jj}(p,p) = H_{pj}(p,j),$$

which is equivalent to

$$\sum_{r=1}^{n}[A_p(j,r)]^2 = \sum_{r=1}^{n}[A_j(p,r)]^2 = \frac{1}{2}\sum_{r=1}^{n}[A_p(p,r)A_j(j,r) + A_j(p,r)A_p(j,r)].$$

From (12), we have

$$\sum_{r=1}^{n}[A_p(j,r)]^2 = \sum_{r=1}^{n}[A_j(p,r)]^2 = \frac{1}{2}\sum_{r=1}^{n} A_j(p,r)A_p(j,r).$$

So

$$0 = 2\sum_{r=1}^{n}[A_p(j,r)]^2 + 2\sum_{r=1}^{n}[A_j(p,r)]^2 - 2\sum_{r=1}^{n} A_j(p,r)A_p(j,r)$$

$$= \sum_{r=1}^{n}[A_p(j,r)]^2 + \sum_{r=1}^{n}[A_j(p,r)]^2 + \sum_{r=1}^{n}[A_p(j,r) - A_j(p,r)]^2,$$

which implies $A_p(j,r) = 0$ for $p,j,r = 1,\cdots,n$.

So $\Delta = 0$ and the theorem is proved.

Our method of proof is still valid in the situation where all the entries in the matrix $\Delta\Delta^T - \frac{1}{2}H(\eta_4)$ are homogeneous polynomials of degree two independent of $n-k$ variables $x_{k+1}, \cdots, x_n$, and we can get the following corollary.

Corollary 1. *Suppose that all the hypotheses about η_4 and Δ in Main Theorem hold except for expression* (1), *which is replaced by the following weaker one that all the entries in the matrix*

$$\Delta\Delta^T - \frac{1}{2}H(\eta_4)$$

are homogeneous polynomials of degree two in $k(k<n)$ variables $x_1,\cdots,x_k$.

Then $\Delta(p,j) = 0, \quad \forall p,j = k+1,\cdots,n.$

Proof. In this situation we have

$$A_pA_j + A_jA_p = -2H_{pj}, \qquad \text{for} \quad p > k \text{ or } j > k,$$

$$A_j^2 = -H_{jj}, \qquad \text{for } j > k.$$

When both p and j are greater than k, we have kept at least one of the lower indices of $H_{ij}(r,l)$ greater than k in the indices-permuting process of the proof of our main theorem. So the proof is still valid and we can get

$$A_p(j,r) = 0, \qquad \forall p,j = k+1,\cdots,n, \qquad r = 1,\cdots,n.$$

Noting the cyclic condition, we get

$$A_r(p,j) = -A_j(r,p) - A_p(j,r) = A_j(p,r) - A_p(j,r) = 0,$$

$$\forall p,j = k+1,\cdots,n, \qquad r = 1,\cdots,n.$$

So $$\Delta(p,j) = \sum_{r=1}^{n} A_r(p,j)\, x_r = 0, \qquad \forall p,j = k+1,\cdots,n.$$

Using this corollary we can derive Tang's recent result on the constant structure of the Ω-matrix, which is stated as follows.

Theorem 1. *If $\mathscr{E}$ is a finite-dimensional estimation algebra of maximal rank and k is the quadratic rank of $\mathscr{E}$, then ω_{ij} are constants for $k+1 \leqslant i, j \leqslant n$.*

Proof. By computing $[[L_0, D_j], D_l]$, we can get that

$$\sum_{i=1}^{n} \beta_{ji}\beta_{li} - \frac{1}{2}\frac{\partial^2 \eta_4}{\partial x_j \partial x_l}$$

are homogeneous degree-two polynomials depending on only $x_1, \cdots, x_k$ for $1 \leqslant j, l \leqslant n$, where β_{ji} is the homogeneous degree-one part of ω_{ji} and k is the quadratic rank of the estimation algebra. (See ref. [5] for the details of the computations.) Let

$$\Delta = (\beta_{ij}).$$

Because $\frac{\partial \omega_{ij}}{\partial x_l} + \frac{\partial \omega_{li}}{\partial x_j} + \frac{\partial \omega_{jl}}{\partial x_i} = 0$ for $0 \leqslant i, j, l \leqslant n$, the cyclic condition is clearly satisfied. Then we invoke Corollary 1 and get

$$\beta_{ij} = \Delta(i,j) = 0, \qquad \text{for } k+1 \leqslant i, j \leqslant n,$$

which completes our proof.

Remark. Tang[5] initially proved theorem 1 by introducing a series of new computations about the estimation algebras, and he first obtained the two key intermediate equations (6) and (11) through computations. Not only have we given a positive answer to an open problem under a very natural condition, but we can also show that Tang's recent result (Theorem 1) can be derived from merely computing $[[L_0, D_j], D_l]$.

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References

1 Wong, W. S., Theorems on the structure of finite dimensional estimation algebras, *Systems Control Lett.*, 1987, 9: 117.

2 Yau S. S. T., Finite dimensional filters with nonlinear drift Ⅰ: A class of filters including both Kalman-Bucy filters and Benes filters, *Journal of Mathematical Systems, Estimation and Control*, 1994, 4: 181.

3 Chen, J., Yau, S. S. T., Finite-dimensional filters with nonlinear drift Ⅵ: Linear structure of Ω, *Mathematics of Control, Signals and Systems*, 1996, 9: 370.

4 Chen, J., Yau, S. S. T., Leung, C.W., Finite-dimensional filters with nonlinear drift Ⅷ: Classification of finite-dimensional estimation algebras of maximal rank with state-space dimension 4, *SIAM J. Control and Optim.*, 1997, 35(4): 1132.

5 Tang, S., Recent advances in classification of finite-dimensional estimation algebras for nonlinear filtering systems, *Progress in Natural Science*, 2000, 10(4): 241.